AD-A037 671 DELAWARE UNIV NEWARK DEPT OF MATHEMATICS F/G 12/1
CONSTRUCTIVE FUNCTION THEORETIC METHODS FOR HIGHER ORDER PSEUDO--ETC(U)
1976 R P GILBERT, G C HSIAO AF-AFOSR-2879-76
UNCLASSIFIED AFOSR-TR-77-0189 NL

1 OF 1
ADA037671

END
DATE
FILMED
4-77

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ADA037671

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFOSR - TR - 77 - 0189 · 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CONSTRUCTIVE FUNCTION THEORETIC METHODS FOR HIGHER ORDER PSEUDOPARABOLIC EQUATIONS	5. TYPE OF REPORT & PERIOD COVERED Interim rept.
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) ~~R.~~ Robert P. Gilbert and G. C. Hsiao	8. CONTRACT OR GRANT NUMBER(s) AFOSR ~~75~~-2879-76
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Delaware Mathematics Department Newark, Delaware 19711	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A4
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling Air Force Base, Washington, D.C. 20332	12. REPORT DATE 1976
	13. NUMBER OF PAGES 17
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

FUNCTION THEORETIC METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS, DARMSTADT -- LECTURE NOTES IN MATHEMATICS, VOL. 561, SPRINGER-VERLAG, BERLIN, 1976.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Riemann function, pseudoparabolic, analytic differential equation, fundamental solutions.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper the approach to higher order, elliptic partial differential equations in the plane is extended to the cases of meta and pseudoparabolic equations. The method makes use of a generalization of the RIEMANN function associated with the complex hyperbolic equation obtained by the analytic continuation $(x,y) \longrightarrow (z,z^*)$. A fundamental solution is obtained whose

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE 404 282

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT cont'd

Taylor (Laurent) expansion in powers of the time has as its first term the classical fundamental solution of VEKUA. Integral representations are obtained for the initial boundary value problems.

2

Reprint from Function Theoretic Methods for Partial Differential Equations, Darmstadt 1976, Lecture Notes in Mathematics, Vol. 561, Springer-Verlag, Berlin, Heidelberg, New York.

CONSTRUCTIVE FUNCTION THEORETIC METHODS FOR HIGHER ORDER PSEUDOPARABOLIC EQUATIONS[‡]

R.P. Gilbert and G.C. Hsiao

0. INTRODUCTION

In this work we will develop a constructive method for solving pseudoparabolic equations of order $2n$ in the plane. More precisely, we investigate equations of the form

$$\underset{\sim}{\mathcal{L}}[u] := \underset{\sim}{M}[u_t] + \underset{\sim}{L}[u] \quad , \tag{0.1}$$

where $\underset{\sim}{M}$ and $\underset{\sim}{L}$ are the respective elliptic operators

$$\underset{\sim}{M}[u] := \Delta^n u + \sum_{k=1}^{n} \underset{\sim}{M}_k(\Delta^{n-k}u) \quad , \tag{0.2}$$

$$\underset{\sim}{M}_k[\phi] := \sum_{p,q=o}^{p+q\leq k} a_k^{pq}(x,y) \frac{\partial^{p+q}\phi}{\partial x^p \partial y^q} \quad ,$$

and

$$\underset{\sim}{L}[u] := \Delta^m u + \sum_{k=1}^{m} \underset{\sim}{L}_k(\Delta^{m-k}u) \quad , \quad m < n \ , \tag{0.3}$$

$$\underset{\sim}{L}_k[\psi] := \sum_{p,q=o}^{p+q\leq k} b_k^{pq}(x,y) \frac{\partial^{p+q}\psi}{\partial x^p \partial y^q} \quad .$$

The coefficients of $\underset{\sim}{M}_k$, $\underset{\sim}{L}_k$ are taken, furthermore, to be analytic functions of x and y for $(x,y) \in D \subset C^1$.

Recently, the integral operator methods of BERGMAN [2] and VEKUA [22], which have been very successful for developing representations for solutions of elliptic equations in the plane, have been extended by COLTON to treat the cases of parabolic [9] and second order pseudoparabolic equations [10], [11] in the plane. BROWN, GILBERT and HSIAO [8] and BROWN and GILBERT [7] developed analogous techniques for fourth-order pseudoparabolic equations using

---

‡ This research was supported in part by the U.S. Air Force Office of Scientific Research through AF-AFOSR Grant No. 76-2879, and in part by the Alexander von Humboldt Foundation.

respectively the methods of VEKUA and BERGMAN to treat elliptic operators. BROWN [6], on the other hand, completed the study of fourth-order, analytic, parabolic equations in two space variables.

Investigations concerning integral operators which generate solutions to parabolic and pseudoparabolic equations in three and four space dimensions have been made by RUNDELL [12], RUNDELL and STECHER [13], and by BHATNAGAR and GILBERT [3], [4], [5].

Pseudoparabolic equations arise in a variety of physical problems, such as the velocity of a non-steady flow of a viscous fluid [21], the theory of seepage of homogeneous fluids through fissured rock [1], hydrostatic excess pressure during the consolidation of clay [19], and the stability of liquid filled shells [18], [23], [24]. GILBERT and ROACH are presently investigating the last mentioned problem as an application of some of the ideas presented in this current work.

I. THE FUNDAMENTAL SOLUTION

If the coefficients of $\underset{\sim}{M}_k$ as analytic functions of x,y have an analytic extension to $(z,z^*) \in D \times D^*$, $D^* := \{z : \bar{z} \in D\}$, where $z = x + iy$, $z^* = x - iy$, then $\underset{\sim}{M}[u]$ has a representation

$$(1.1) \quad \underset{\sim}{M}[u] := \sum_{k,j=0}^{n} A_{kj}(z,z^*) \frac{\partial^{k+j}U}{\partial z^k \partial z^{*j}}, \quad U(z,z^*) := u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right),$$

with $A_{nn} \equiv 1$. We assume also that the operator $\underset{\sim}{L}$ has a complex form

$$(1.2) \quad \underset{\sim}{L}[U] := \sum_{k,j=0}^{n-1} B_{kj}(z,z^*) \frac{\partial^{k+j}U}{\partial z^k \partial z^{*j}}.$$

The adjoint operator to $\underset{\sim}{\mathcal{L}}$ is given by $\underset{\sim}{\mathcal{L}}^* = \underset{\sim}{\mathcal{M}}$,

$$(1.3) \quad \underset{\sim}{\mathcal{M}}[U] := \underset{\sim}{M}^*[U_t] - \underset{\sim}{L}^*[U],$$

with

$$(1.4) \quad \underset{\sim}{M}^*[U] := \sum_{k,j=0}^{n} (-1)^{k+j} \frac{\partial^{k+j}}{\partial z^k \partial z^{*j}} (A_{kj}U),$$

$$(1.5) \quad \underset{\sim}{L}^*[U] := \sum_{k,j=0}^{n-1} (-1)^{k+j} \frac{\partial^{k+j}}{\partial z^k \partial z^{*j}} (B_{kj}U).$$

A function S of the form

(1.6) $$S(x,y,t;\xi,\eta,\tau) := A(x,y,t;\xi,\eta,\tau)\ \ln \frac{1}{r} + B(x,y,t;\xi,\eta,\tau)$$

where $r = [(x-\xi)^2 + (y-\eta)^2]^{1/2}$ will be called a fundamental solution of (1.1) if it satisfies the following conditions

(c-1) As a function of (x,y,t), S is a solution of the adjoint equation $\underset{\sim}{\mathcal{M}}[S] = 0$ and is an analytic function of its argument except at $r = 0$, where $\Delta^{n-1}S$ has a logarithmic singularity.

(c-2) At the parameter point $x = \xi$, $y = \eta$ we have
$$\frac{\partial^{p+q}A_t}{\partial x^p \partial y^q} = 0 \quad \text{for} \quad p+q \le 2n-3\ , \quad \text{and} \quad \Delta^{n-1}A_t = 4^{n-1}.$$

(c-3) A and B are analytic functions of (x,y,t) at $r = 0$ and vanish at $t = \tau$.

Remark: The above implies that A may be written as $r^{2n-2}\hat{A}(x,y,t;\xi,\eta,\tau)$ with $\hat{A}$ regular at the parameter point. The above notation is computationally easy to work with.

We intend to show that it is possible to develop the coefficients A and B as analytic functions with the expansions

(1.7) $$A(z,z^*,t;\zeta,\zeta^*,\tau) = \sum_{j=1}^{\infty} A_j(z,z^*;\zeta,\zeta^*)\ \frac{(t-\tau)^j}{j!}\ ,$$

(1.8) $$B(z,z^*,t;\zeta,\zeta^*,\tau) = \sum_{j=1}^{\infty} B_j(z,z^*;\zeta,\zeta^*)\ \frac{(t-\tau)^j}{j!}\ .$$

Furthermore, we shall identify $A_1(z,z^*;\zeta,\zeta^*)$ as the Riemann function corresponding to the operator $\underset{\sim}{M}$. The other coefficients will be seen to satisfy homogeneous Goursat conditions on $z = \zeta$ and $z^* = \zeta^*$.

Inserting (1.6) into $\underset{\sim}{\mathcal{M}}[U] := \underset{\sim}{M}^*[U_t] - \underset{\sim}{L}^*[U] = 0$ we obtain, after some manipulation of terms

(1.9) $$\underset{\sim}{\mathcal{M}}[S] \equiv \underset{\sim}{\mathcal{M}}[A]\ \ln \frac{1}{r} + \underset{\sim}{\mathcal{M}}[B] + I_n + I_n^* = 0$$

with

$$I_n := \left\{ \frac{1}{2}(n-1)! \sum_{j=o}^{n} (-1)^j \frac{\partial^j}{\partial z^{*j}} (A_{nj} A_t) \right\} (z-\zeta)^{-n}$$

$$(1.10) \qquad + \frac{1}{2} \sum_{p=1}^{n-1} \left\{ \binom{n}{n-p} \sum_{j=o}^{n} (-1)^{n+j} \frac{\partial^{n+j-p}}{\partial z^{n-p} \partial z^{*j}} (A_{nj} A_t) \right.$$

$$+ \sum_{k=p}^{n-1} \binom{k}{k-p} \left[(-1)^{k+n} \frac{\partial^{k+n-p}}{\partial z^{k-p} \partial z^{*n}} (A_{kn} A_t) \right.$$

$$\left. \left. + \sum_{j=o}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (A_{kj} A_t - B_{kj} A) \right] \right\} \frac{(-1)^p (p-1)!}{(z-\zeta)^p}$$

and

$$I_n^* := \left\{ \frac{1}{2}(n-1)! \sum_{k=o}^{n} (-1)^k \frac{\partial^k}{\partial z^k} (A_{kn} A_t) \right\} (z^*-\zeta^*)^{-n}$$

$$(1.11) \qquad + \frac{1}{2} \sum_{q=1}^{n-1} \binom{n}{n-q} \sum_{k=o}^{n} (-1)^{k+n} \frac{\partial^{k+n-q}}{\partial z^k \partial z^{*n-q}} (A_{kn} A_t)$$

$$+ \sum_{j=q}^{n-1} \left\{ \binom{j}{j-q} \left[(-1)^{n+j} \frac{\partial^{n+j-q}}{\partial z^n \partial z^{*j-q}} (A_{nj} A_t) \right. \right.$$

$$\left. \left. + \sum_{k=o}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} (A_{kj} A_t - B_{kj} A) \right] \right\} \frac{(-1)^q (q-1)!}{(z^*-\zeta^*)^q} .$$

Because of the multivaluedness of the logarithmic singularity it is necessary to set $\underset{\sim}{\mathcal{M}}[A] = 0$. To cancel the poles at $z = \zeta$ and $z^* = \zeta^*$ we ask that the coefficients of $(z-\zeta)^p$, $(z^*-\zeta^*)^p$, $(p=1,\ldots,n)$ vanish. These latter conditions provide us with so-called Goursat data for the A_j, B_j coefficients in the representations (1.9). Setting (1.7) into (1.10) and (1.11) we obtain the following conditions which $A_1(z,z^*;\zeta,\zeta^*)$ must satisfy

$$(1.12) \qquad \sum_{k=p}^{n} \binom{k}{p} \sum_{j=o}^{n} (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} \left[A_{kj}(\zeta,z^*) A_1(\zeta,z^*;\zeta,\zeta^*) \right] = 0$$

$$(p=1,2,\ldots,n), \quad z = \zeta ,$$

and

$$(1.13) \qquad \sum_{j=q}^{n} \binom{j}{q} \sum_{k=o}^{n} (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} \left[A_{kj}(z,\zeta^*)\ A_1(z,\zeta^*;\zeta,\zeta^*) \right] = 0$$

$$(q=1,2,\ldots,n), \quad z^* = \zeta^* .$$

We recall that condition (c-2) implies that

$$(1.14) \qquad \frac{\partial^{p+q}}{\partial z^p \partial z^{*q}} A_j(\zeta,\zeta^*;\zeta,\zeta^*) = 0 , \quad \text{for } j=1,2,3,\ldots ,$$

$$(p,q=0,1,2,\ldots,2n-3 , \text{ with } p+q \leq 2n-3) .$$

In order to show that A_1 is actually the Riemann function, it is sufficient for us to show that the above conditions are equivalent to the characteristic conditions which uniquely determine it. We recall from VEKUA [22], Chapter V, the following conditions imposed on the Riemann function $R(z,z^*;\zeta,\zeta^*)$ and for convenience we label these conditions using his equation numbers :

$$(37.28) \qquad \underset{\sim}{M}^*[R] = 0$$

$$(37.29) \qquad \frac{\partial^k R}{\partial z^k}(z,z^*;\zeta,\zeta^*)\Big|_{z=\zeta} = 0 , \quad \frac{\partial^k R}{\partial \zeta^k}(z,z^*;\zeta,\zeta^*)\Big|_{z^*=\zeta^*} = 0 ,$$

$$(k=0,1,2,\ldots,n-2)$$

$$(37.30) \qquad \begin{aligned} &\frac{\partial^{n-1} R}{\partial z^{n-1}}(z,z^*;\zeta,\zeta^*)\Big|_{z=\zeta} = X(z^*,\zeta^*,\zeta) , \\ &\frac{\partial^{n-1} R}{\partial z^{*n-1}}(z,z^*;\zeta,\zeta^*)\Big|_{z^*=\zeta^*} = X^*(z,\zeta,\zeta^*) . \end{aligned}$$

Here X, and X* are solutions respectively of the ordinary differential equations

$$(37.31) \qquad \begin{aligned} &\sum_{k=o}^{n} (-1)^k \frac{\partial^k}{\partial z^{*k}} \left[A_{nk}(\zeta,z^*)\ X \right] = 0 , \quad \text{and} \\ &\sum_{k=o}^{n} (-1)^k \frac{\partial^k}{\partial z^k} \left[A_{kn}(z,\zeta^*)\ X^* \right] = 0 . \end{aligned}$$

Furthermore, X and X* are seen to satisfy the intial conditions

$$(37.32)\qquad \frac{\partial^k X}{\partial z^{*k}}(z^*,\zeta^*,\zeta)\Big|_{z^*=\zeta^*} = 0\ ,\quad \frac{\partial^{n-1} X}{\partial z^{*n-1}}(z^*,\zeta^*,\zeta)\Big|_{z^*=\zeta^*} = 1\ ,$$

$$(k=0,1,\ldots n-2)$$

$$\frac{\partial^k X^*}{\partial z^k}(z,\zeta,\zeta^*)\Big|_{z=\zeta} = 0\ ,\quad \frac{\partial^{n-1} X^*}{\partial z^{n-1}}(z,\zeta,\zeta^*)\Big|_{z=\zeta} = 1\ .$$

$$(k=o,1,\ldots n-2)$$

We note that $A_1(z,z^*;\zeta,\zeta^*)$ automatically satisfies (37.28), (37.29), (37.32) by virtue of our conditions (c-1) and (c-2). This suggests that we check to see if the remaining conditions (37.30), (37.31) are compatible with ours. Using (37.29) in combination with condition (1.12) we obtain for $p > 1$

$$(1.15)\qquad \sum_{j=o}^{n}(-1)^{n+j}\frac{\partial^{n+j-p}}{\partial z^{n-p}\partial z^{*j}}\Big[A_{nj}(z,z^*)\ A_1(z,z^*;\zeta,\zeta^*)\Big]\Big|_{z=\zeta} \equiv 0\ .$$

In the case where $p = 1$, this becomes

$$(1.16)\qquad \begin{aligned} &\sum_{j=o}^{n}(-1)^{n+j}\sum_{\ell=o}^{n-1}\binom{n-1}{\ell}\frac{\partial^j}{\partial z^{*j}}\Big[\frac{\partial^{n-1-\ell}}{\partial z^{n-1-\ell}}A_{nj}(z,z^*)\ \frac{\partial^{\ell}A_1}{\partial z^{\ell}}(z,z^*;\zeta,\zeta^*)\Big]\Big|_{z=\zeta} \\ &= \sum_{j=o}^{n}(-1)^{n+j}\frac{\partial^j}{\partial z^{*j}}\Big[A_{nj}(\zeta,z^*)\ \frac{\partial^{n-1}A_1}{\partial z^{n-1}}(z,z^*;\zeta,\zeta^*)\Big|_{z=\zeta}\Big] = 0\ . \end{aligned}$$

Identifying temporarily A_1 with R and identifying the $(n-1)^{st}$ derivative with respect to z as X as given in (37.30), (1.16) becomes the first of equations (37.31). Repeating this analysis with (1.13) and identifying X^* as the $(n-1)^{st}$ derivative with respect to z^*, (37.30) yields likewise the second equation of (37.31). This exhausts our conditions and leaves us free to impose the additional initial conditions on X and X^* prescribed by (37.32). We conclude that we are permitted to identify $A_1 \equiv R$.

Lemma: The first coefficient $A_1(z,z^*;\zeta,\zeta^*)$ for A in the representation (1.7) may be taken as the Riemann function associated with $\underset{\sim}{M}[u] = 0$. The coefficients $A_p(z,z^*;\zeta,\zeta^*)$, $p \geq 2$, are determined recursively by the nonhomogeneous equations

$$\underset{\sim}{M}^*[A_p] = \underset{\sim}{L}^*[A_{p-1}] \quad , \tag{1.17}$$

and the homogeneous Goursat data,

$$\begin{aligned} &\frac{\partial^\ell}{\partial z^\ell} A_p(z,z^*;\zeta,\zeta^*)\Big|_{z=\zeta} = 0 \ , \qquad \ell=0,1,\ldots,n-1 \ , \\ &\frac{\partial^\ell}{\partial z^{*\ell}} A_p(z,z^*;\zeta,\zeta^*)\Big|_{z^*=\zeta^*} = 0 \ , \qquad \ell=0,1,\ldots,n-1 \ . \end{aligned} \tag{1.18}$$

Proof: Since $S = A \log \frac{1}{r} + B$ satisfies the adjoint equation, it follows that $\underset{\sim}{\mathcal{M}}[A] = 0$. This in turn implies $\underset{\sim}{M}^*[A_1] = 0$, and $\underset{\sim}{M}^*[A_{j+1}] = \underset{\sim}{L}^*[A_j]$, $(j=0,1,\ldots)$. A moment's reflection concerning our condition (c-1) indicates that it can be satisfied using the above conditions (1.18), when A_1 is taken to be the Riemann function for $\underset{\sim}{M}[U] = 0$.

II. DETERMINATION OF THE COEFFICIENTS $A_p(z,z^*;\zeta,\zeta^*)$

The series representations (1.9) for the coefficients A and B of the singular solution

$$S(z,z^*,t;\zeta,\zeta^*,\tau) := A(z,z^*,t;\zeta,\zeta^*,\tau)\ \ell n \frac{1}{r} + B(z,z^*,t;\zeta,\zeta^*,\tau)$$

suggest that we try to determine A_p and B_p successively. To this end we develop a Green's formula based on the formal identity

$$V\,\underset{\sim}{M}[U] - U\,\underset{\sim}{M}^*[v] \equiv \frac{\partial P}{\partial z}(z,z^*) + \frac{\partial P^*}{\partial z^*}(z,z^*) \ , \tag{2.1}$$

where $\underset{\sim}{M}$ and $\underset{\sim}{M}^*$ are given by (1.4) and (1.7 and

$$P(z,z^*) := \sum_{\substack{j=o\\k=1}}^{n} \sum_{p=o}^{k-1} (-1)^p \frac{\partial^p}{\partial z^p}(VA_{kj}) \frac{\partial^{k+j-p-1}U}{\partial z^{k-p-1}\partial z^{*j}} \ , \tag{2.2}$$

and

$$P^*(z,z^*) := \sum_{\substack{k=o\\j=1}}^{n} \sum_{q=o}^{j-1} (-1)^{k+q} \frac{\partial^{q+k}}{\partial z^{*q}\partial z^k}(VA_{kj}) \frac{\partial^{j-q-1}U}{\partial z^{*j-q-1}} \ .$$

If $\underset{\sim}{M}_s$ is meant as the $\underset{\sim}{M}$ operator with s, s^* replacing z, z^*, then, setting $U(s,s^*) := A_1(z,z^*;s,s^*)$ and noting that A_1 is also a solution of $\underset{\sim}{M}_s[U] = 0$, the identity (2.1) yields

$$(2.4) \qquad - U(s,s^*)\, \underset{\sim}{M}^*_s[V] \equiv \frac{\partial P}{\partial s}(s,s^*) + \frac{\partial P^*}{\partial s^*}(s,s^*) \quad .$$

Lemma 2: The coefficients $A_{p+1}(z,z^*;\zeta,\zeta^*)$ may be formally determined by the recursive scheme

$$(2.5) \qquad A_{p+1}(z,z^*;\zeta,\zeta^*) = \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^{z} ds\, A_1(z,z^*;s,s^*)\, \underset{\sim}{L}^*[A_p(s,s^*;\zeta,\zeta^*)] \, , \quad p=1,2,\ldots \quad .$$

Proof: Integrating (2.4) yields

$$- \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^{z} ds\, U(s,s^*)\, \underset{\sim}{M}^*[V] = \int_{\zeta^*}^{z} P(z,s^*)ds^* + \int_{\zeta}^{z} P^*(s,z^*)ds - Q(z,z^*;\zeta,\zeta^*) \, ,$$

with

$$Q(z,z^*;\zeta,\zeta^*) = \int_{\zeta^*}^{z^*} P(\zeta,s^*)ds^* - \int_{\zeta}^{z} P^*(s,\zeta^*)ds \quad .$$

Using the conditions (37.29) and recalling $U := A_1$, we can simplify the integral of P above as

$$(2.6) \qquad \int_{\zeta^*}^{z^*} P(z,s^*)ds^* = \sum_{j=o}^{n} \int_{\zeta^*}^{z^*} V(z,s^*)\, A_{nj}(z,s^*)\, \frac{\partial^{n-1+j} U(z,s^*)}{\partial z^{n-1} \partial s^{*j}}\, ds^* \, .$$

According to VEKUA [22] we may identify the function

$$(37.10) \qquad g(s^*;z,z^*) := \frac{\partial^{n-1}}{\partial z^{n-1}} A_1(z,z^*;z,s^*) \quad ,$$

which, furthermore, satisfies an ordinary differential equation

$$(37.9) \qquad \frac{d^n g}{d\zeta^n} + \sum_{m=o}^{n-1} A_{nm}(z,\zeta)\, \frac{d^m g}{d\zeta^m} = 0 \quad .$$

Using (37.9), the right-hand side of (2.6) is seen to vanish identically.

We next turn our attention to the P^*-integral, and note that (37.29) implies that $\frac{\partial^{j-q-1}U}{\partial z^{*\,j-q-1}}(s,z^*) = 0$, except for $q = 0$, $j = n$. Hence

$$\int_\zeta^z P^*(s,z^*)ds = \sum_{k=o}^{n} (-1)^k \int_\zeta^z \left\{ \frac{\partial^k}{\partial s^k}\left[V(s,z^*)\ A_{kn}(s,z^*)\right] \cdot \frac{\partial^{n-1}\ U(s,z^*)}{\partial z^{*\,n-1}} \right\} ds \quad . \tag{2.7}$$

The associated functions

$$g^*(s;z,z^*) := \frac{\partial^{n-1}}{\partial s^{*\,n-1}} A_1(z,z^*;s,s^*)\Big|_{s^*=z^*} \tag{37.10}$$

are known [24] to satisfy the ordinary differential equation

$$\frac{d^n g^*}{ds^n} + \sum_{m=o}^{n-1} A_{mn}(s,z^*) \frac{d^m g^*}{ds^m} = 0 \quad , \tag{37.9}$$

and the initial conditions

$$\begin{aligned} &\frac{\partial^\ell g^*}{\partial s^\ell}(s;z,z^*)\Big|_{s=z} = 0 \ , \qquad (\ell=0,1,\ldots,n-2) \\ &\frac{\partial^{n-1} g^*}{\partial s^{n-1}}(s;z,z^*)\Big|_{s=z} = 1 \quad . \end{aligned} \tag{37.11}$$

Consequently, after some regrouping, it may be seen that

$$\begin{aligned} \int_\zeta^z P^*(s,z^*)ds &= \sum_{k=1}^{n} \int_\zeta^z \frac{\partial}{\partial s}\left\{ \sum_{\ell=o}^{k-1} (-1)^{\ell+k} \frac{\partial^{k-1-\ell}}{\partial s^{k-1-\ell}}\left[V(s,z^*)\ A_{kn}(s,z^*)\right] \cdot \frac{\partial^\ell g^*(s;z,z^*)}{\partial s^\ell} \right\} ds \\ &= (-1)^{2n-1}\, V(z,z^*)\ A_{nn}(z,z^*) \frac{\partial^{n-1}}{\partial z^{n-1}} g^*(z;z,z^*) + Q^*(\zeta;z,z^*) \\ &= -\ V(z,z^*) + Q^*(\zeta;z,z^*) \quad , \end{aligned} \tag{2.8}$$

where

$$Q^*(\zeta;z,z^*) := -\sum_{k=1}^{n} \sum_{\ell=o}^{k-1} (-1)^{\ell+k} \frac{\partial^{k-1-\ell}}{\partial s^{k-1-\ell}}\left[V(\zeta,z^*)\ A_{kn}(\zeta,z^*)\right] \cdot \frac{\partial^\ell g^*(\zeta;z,z^*)}{\partial \zeta^\ell} \quad . \tag{2.9}$$

Putting together the above terms we note that the following representation for $V(z,z^*)$ has been obtained

$$V(z,z^*) = Q^*(\zeta;z,z^*) - Q(z,z^*;\zeta,\zeta^*)$$

$$(2.10) \qquad + \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^{z} ds\, A_1(z,z^*;s,s^*)\, \underset{\sim}{M}^*[V(s,s^*)] \; .$$

Setting $V(z,z^*) := A_{p+1}(z,z^*;\zeta,\zeta^*)$ and recalling the initial conditions (1.18), we conclude $Q^*(\zeta;z,z^*) \equiv 0$. The expression for $Q(z,z^*;\zeta,\zeta^*)$ may also be seen to vanish identically by virtue of the identities

$$\int_{\zeta^*}^{z^*} P(\zeta,s^*)ds^* = \sum_{\substack{j=o\\k=1}}^{n} \sum_{\ell=o}^{k-1} (-1)^{\ell} \int_{\zeta^*}^{z^*} ds^* \frac{\partial^{\ell}}{\partial s^{\ell}} \left[A_{p+1}(\zeta,s^*;\zeta,\zeta^*)\, A_{kj}(\zeta,s^*)\right] \cdot \frac{\partial^{k+j-\ell-1}}{\partial z^{k-\ell-1}\partial s^{*j}}\, U(\zeta,s^*) \equiv 0 \quad ,$$

and

$$\int_{\zeta}^{z} P^*(s,\zeta^*)ds = \sum_{\substack{k=o\\j=1}}^{n} \sum_{q=o}^{j-1} (-1)^{k+q} \int_{\zeta}^{z} \frac{\partial^{q+k}}{\partial z^{*q}\partial s^{k}} \left(A_{p+1}(s,\zeta^*;\zeta,\zeta^*) \cdot A_{kj}(s,\zeta^*)\right) \frac{\partial^{j-q-1}}{\partial z^{*j-q-1}}\, U(s,\zeta^*)ds \equiv 0 \quad .$$

Our Lemma is proved at this point by recognizing that $\underset{\sim}{M}^*[A_{p+1}) = \underset{\sim}{L}^*(A_p)$.

<u>Lemma 3</u>: <u>The coefficients</u> $A_{p+1}(z,z^*;\zeta,\zeta^*)$ <u>may be computed recursively by</u>

$$(2.11) \quad A_{p+1}(z,z^*;\zeta,\zeta^*) = \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^{z} ds\, A_p(z,z^*;s,s^*)\, \underset{\sim}{L}^*[A_1(s,s^*;\zeta,\zeta^*)] \; ,$$

$p=1,2,\ldots$. <u>Furthermore, the series representation for</u> $A(z,z^*,t;\zeta,\zeta^*,\tau)$ <u>converges uniformly in the domain</u> $(G \times G^* \times T)^2$ <u>where</u> G <u>is the domain of regularity of the coefficients in the</u> x-y <u>coordinates and</u> T <u>is a disk in the complex t-plane</u>.

<u>Proof:</u> Recalling the form of $\underset{\sim}{L}$ and $\underset{\sim}{L}^*$, we note that we may write

$$A_1(z,z^*;s,s^*)\ \underset{\sim}{L}^*[A_p(s,s^*;\zeta,\zeta^*)] - A_p(s,s^*;\zeta,\zeta^*)\ \underset{\sim}{L}[A_1(z,z^*;s,s^*)]$$

$$= -\sum_{\substack{j=o\\k=1}}^{n-1} \frac{\partial}{\partial s}\left\{\sum_{\ell=o}^{k-1}(-1)^{\ell}\frac{\partial^{\ell}}{\partial s^{\ell}}\left[A_p(s,s^*;\zeta,\zeta^*)\ B_{kj}(s,s^*)\right]\frac{\partial^{k+j-\ell-1}}{\partial s^{k-\ell-1}\partial s^{*j}}A_1(z,z^*;s,s^*)\right\}$$

$$-\sum_{\substack{k=o\\j=1}}^{n-1}(-1)^k\frac{\partial}{\partial s^*}\left\{\sum_{m=o}^{j-1}(-1)^m\frac{\partial^{m+k}}{\partial s^{*m}\partial s^k}\left[A_p(s,s^*;\zeta,\zeta^*)B_{kj}(s,s^*)\right]\frac{\partial^{j-m-1}}{\partial s^{*j-m-1}}A_1(z,z^*;s,s^*)\right\}$$

Integration gives the following identity

$$\int_{\zeta^*}^{z^*}ds^*\int_{\zeta}^{z}ds\ A_1(z,z^*;s,s^*)\ \underset{\sim}{L}^*[A_p(s,s^*;\zeta,\zeta^*)]$$
$$= \int_{\zeta^*}^{z^*}ds^*\int_{\zeta}^{z}ds\ A_p(s,s^*;\zeta,\zeta^*)\ \underset{\sim}{L}[A_1(z,z^*;s,s^*)]$$
$$+ H + H^*\ ,$$

where

$$H := -\sum_{\substack{j=o\\k=1}}^{n-1}\sum_{\ell=o}^{k-1}(-1)^{\ell}\int_{\zeta^*}^{z^*}ds^*\left\{\frac{\partial^{\ell}}{\partial s^{\ell}}\left[A_p(s,s^*;\zeta,\zeta^*)\ B_{kj}(s,s^*)\right]\ \frac{\partial^{k+j-\ell-1}}{\partial s^{k-\ell-1}\partial s^{*j}}\right.$$

(2.12)

$$\left.\cdot\ A(z,z^*;s,s^*)\right\}_{s=\zeta}^{s=z}$$

and

$$H^* := \sum_{\substack{k=o\\j=1}}^{n-1}(-1)^k\sum_{m=o}^{j-1}(-1)^m\int_{\zeta}^{z}ds\left\{\frac{\partial^{m+k}}{\partial s^{*m}\partial s^k}\left[A_p(s,s^*;\zeta,\zeta^*)\ B_{kj}(s,s^*)\right]\right.$$

(2.13)

$$\left.\cdot\ \frac{\partial^{j-m-1}}{\partial s^{*j-m-1}}A_1(z,z^*;s,s^*)\right\}_{s^*=\zeta^*}^{s^*=z^*}\ .$$

Using the conditions (1.18) for A_{p+1} and (37.29) for A_1, the terms H and H^* are seen to identically vanish. This establishes the recursive definition (2.11). To show that the series for A converges we note that according to VEKUA [22], p. 186 $A_1(z,z^*;\zeta,\zeta^*)$ is dominated by

$$\frac{|z-\zeta|^{n-1}|z^*-\ {}^*|^{n-1}}{[(n-1)!]^2}\quad \text{in}\quad [G\times G^*]^2\ .$$

Consequently, using (2.11) it follows by induction that

$$(2.14) \qquad |A_p(z,z^*;\zeta,\zeta^*)| \le C \frac{|z-\zeta|^{n+p-2}|z^*-\zeta^*|^{n+p-2}}{[(n+p-2)!]^2} \, \| \underset{\sim}{L}[A_1] \|^{p-1}$$

where

$$\| \underset{\sim}{L}[A_1] \| := \sup_{(G \times G^*)^2} |\underset{\sim}{L}[A_1]|$$

and

$$C := \sup_{(G \times G^*)^2} |A_1| \quad .$$

From (2.14) it is clear that the series for A converges in the stated domain.

III. DETERMINATION OF $B(z,z^*,t;\zeta,\zeta^*,\tau)$

From (1.10) we obtain as the partial differential equation for B,

$$(3.1) \qquad \underset{\sim}{\mathcal{M}}[B] = - I_n - I_n^* \, ,$$

where I_n and I_n^* are given by (1.11) and (1.12) respectively. Putting the series expansion (1.8) for B into the left-hand side of (3.1) yields then a recursive scheme for the coefficients B_j, namely

$$(3.2) \qquad \underset{\sim}{M}^*[B_1] = f_o(z,z^*;\zeta,\zeta^*) := -\frac{1}{2} \sum_{p=1}^{n} \frac{(-1)^p (p-1)!}{(z-\zeta)^p} \left\{ \sum_{k=p}^{n} \binom{k}{p} \cdot \sum_{j=o}^{n} (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (A_{kj} A_1) \right\}$$

$$-\frac{1}{2} \sum_{q=1}^{n} \frac{(-1)^q (q-1)!}{(z^*-\zeta^*)^q} \left\{ \sum_{j=q}^{n} \binom{j}{q} \sum_{k=o}^{n} (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^{k} \partial z^{*j-q}} (A_{kj} A_1) \right\} ,$$

$$(3.3) \qquad M^*[B_{\ell+1}] = f_\ell(z,z^*;\zeta,\zeta^*) := L^*[B_\ell] - \frac{1}{2} \frac{(n-1)!}{(z-\zeta)^n} \sum_{j=o}^{n} (-1)^j \frac{\partial^j}{\partial z^{*j}} (A_{nj} A_{\ell+1})$$

$$-\frac{1}{2} \sum_{p=1}^{n-1} \frac{(-1)^p (p-1)!}{(z-\zeta)^p} \left\{ \sum_{k=p}^{n} \binom{k}{p} \sum_{j=0}^{n} (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (A_{kj} A_{\ell+1}) \right.$$

$$\left. - \sum_{k=p}^{n-1} \binom{k}{p} \sum_{j=o}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-p}}{\partial z^{k-p} \partial z^{*j}} (B_{kj} A_\ell) \right\} +$$

$$- \frac{1}{2} \frac{(n-1)!}{(z^*-\zeta^*)^n} \sum_{k=o}^{n} (-1)^k \frac{\partial^k}{\partial z^k} (A_{kn} A_{\ell+1})$$

$$- \frac{1}{2} \sum_{q=1}^{n-1} \frac{(-1)^q (q-1)!}{(z^*-\zeta^*)^q} \left\{ \sum_{j=q}^{n} \binom{j}{q} \sum_{k=o}^{n} (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} (A_{kj} A_{\ell+1}) \right.$$

$$\left. - \sum_{j=q}^{n-1} \binom{j}{q} \sum_{k=o}^{n-1} (-1)^{k+j} \frac{\partial^{k+j-q}}{\partial z^k \partial z^{*j-q}} (B_{kj} A_{\ell}) \right\} ,$$

$\ell=1,2,\ldots$.

If we specify that the B_ℓ satisfy the homogeneous Goursat data

$$(3.4) \qquad \left. \frac{\partial^k B_\ell(z,z^*;\zeta,\zeta^*)}{\partial z^{*k}} \right|_{z^*=\zeta^*} = 0 , \qquad \left. \frac{\partial^k B_\ell(z,z^*;\zeta,\zeta^*)}{\partial z^k} \right|_{z=\zeta} = 0 , \qquad k=0,1,2,\ldots,n-1, \quad \ell=1,2,\ldots ,$$

then

$$B_{\ell+1}(z,z^*;\zeta,\zeta^*) = \int_{\zeta^*}^{z^*} ds^* \int_{\zeta}^{z} ds \; f_\ell(s,s^*;\zeta,\zeta^*) \; A_1(z,z^*;s,s^*) ,$$

$\ell=0,1,\ldots$.

The majoration for the $B_{\ell+1}$, while technically involved, proceeds by the usual methods [10], [11], [8].

We summarize the discussion of Sections II and III in the following theorem.

Theorem 1: Assume that the coefficients A_{jk}, B_{jk} are analytic functions of two complex variables z, z^* in the bicylinder $D \times D^*$. Then $A(z,z^*,t;\zeta,\zeta^*,\tau)$ and $B(z,z^*,t;\zeta,\zeta^*,\tau)$ are analytic functions of their six independent variables for all (complex) t, τ, and $z,\zeta \in D$, $z^*,\zeta^* \in D^*$. Moreover, both A and B can be represented by a uniformly convergent series expansion in the form of (1.7) and (1.8) respectively.

IV. BOUNDARY VALUE PROBLEMS

We now proceed to use the fundamental solution (1.8) to develop representations for boundary value problems. The solutions to boundary value problems we will seek in the class

$$(4.1)\quad \mathcal{G} := \{u(x,y,t) : u \in C^{2n-2}(\bar{D}\times T);\ u_t \in C^{2n}(D\times T)\cap C^{2n-1}(\bar{D}\times T)\},$$

where $T := \{t : 0 \le t < t_o\}$ and where t_o is a fixed constant.

We begin by considering the identity

$$(4.2)\quad \begin{aligned} v_t \underset{\sim}{\mathcal{L}}[u] - u_t \underset{\sim}{\mathcal{M}}[v] &\equiv \{v_t \underset{\sim}{M}[u_t] - u_t \underset{\sim}{M}^*[v_t]\} \\ &- \{v \underset{\sim}{L}[u_t] - u_t \underset{\sim}{L}^*[v]\} + \frac{\partial}{\partial t}\{v \underset{\sim}{L}[u]\} \ . \end{aligned}$$

If we replace v by the fundamental solution

$$S(x,y,t;\xi,\eta,\tau) = A \log\frac{1}{r} + B \ ,$$

then (4.2) leads to an integral representation for solutions which satisfy the homogeneous initial data $u(x,y,o) = 0$ in D, namely

$$(4.3)\quad u(\xi,\eta,\tau) = \frac{1}{2\pi[(n-1)!]^2}\int_0^\tau dt \int_{\partial D} H[u(x,y,t),\ S(x,y,t;\xi,\eta,\tau)] \ .$$

Here H is a bilinear form in $\dfrac{\partial^{k+j}u_t}{\partial x^k \partial y^j}$, $\dfrac{\partial^{k+j}S_t}{\partial x^k \partial y^j}$, $(k+j \le 2n-1)$ as well as $\dfrac{\partial^{k+j}u}{\partial x^k \partial y^j}$, $\dfrac{\partial^{k+j}S}{\partial x^k \partial y^j}$, $(k+j \le 2n-3)$. The introduction of a special, Green-type, fundamental solution, namely one which satisfies on ∂D

$$(4.4)\quad S_t = \frac{\partial S_t}{\partial \nu} = \dots = \frac{\partial^{n-1} S_t}{\partial \nu^{n-1}} = 0 \qquad (\nu = \text{inward normal})$$

permits us to reduce the bilinear form $H[\ ,\ ]$ to the case where only the boundary data of the first kind

$$(4.5)\quad u_t^+ = f_o\ ,\ \frac{\partial u_t^+}{\partial \nu} = f_1\ ,\ \dots\ ,\ \frac{\partial^{n-1} u_t^+}{\partial \nu^{n-1}} = f_n \quad \text{on } \partial D$$

appears.

This representation is more easily computed using the complex notation. To this end we recall the elementary identities,

$$\frac{d}{ds}\left(\frac{\partial^{k+m}u}{\partial z^k \partial z^{*m}}\right)^+ \equiv \left(\frac{\partial^{k+m+1}u}{\partial z^{k+1}\partial z^{*m}}\right)^+ \frac{dz}{ds} + \left(\frac{\partial^{k+m+1}u}{\partial z^k \partial z^{*m+1}}\right)^+ \frac{dz^*}{ds} ,$$

(4.6)

$$\left(\frac{d^k u}{d\nu^k}\right)^+ = i^k \sum_{\ell=o}^{k} (-1)^{k-\ell}\binom{k}{\ell}\left(\frac{\partial^k u}{\partial z^{k-\ell}\partial z^{*\ell}}\right)^+ \left(\frac{dz}{ds}\right)^{k-2\ell} ,$$

which hold on ∂D. The first two brackets of (4.2) we compute directly as

$$\{V_t\ M[U_t] - U_t\ M^*[V_t]\} - \{V\ L[U_t] - U_t\ L^*[V]\} \tag{4.7}$$

$$= \frac{\partial}{\partial z}\tilde{P} + \frac{\partial}{\partial z^*}\tilde{P}^* ,$$

where

$$\tilde{P} := \sum_{\substack{k=1\\ j=o}}^{n} \sum_{\ell=o}^{k-1} (-1)^\ell \frac{\partial^\ell}{\partial z^\ell}(V_t A_{kj}) \frac{\partial^{k+j-\ell-1}U_t}{\partial z^{k-\ell-1}\partial z^{*j}}$$

(4.8)

$$- \sum_{\substack{k=1\\ j=o}}^{n-1} \sum_{\ell=o}^{k-1} (-1)^\ell \frac{\partial^\ell}{\partial z^\ell}(VB_{kj}) \frac{\partial^{k+j-\ell-1}U_t}{\partial z^{k-\ell-1}\partial z^{*j}} ,$$

$$\tilde{P}^* := \sum_{\substack{k=o\\ j=1}}^{n} (-1)^k \sum_{m=o}^{j-1} (-1)^m \frac{\partial^{m+k}}{\partial z^{*m}\partial z^k}(V_t A_{kj}) \frac{\partial^{j-m-1}U_t}{\partial z^m}$$

(4.9)

$$- \sum_{\substack{k=o\\ j=1}}^{n-1} (-1)^k \sum_{m=o}^{j-1} (-1)^m \frac{\partial^{m+k}}{\partial z^{*m}\partial z^k}(VB_{kj}) \frac{\partial^{j-m-1}U_t}{\partial z^m} .$$

Then (4.3) takes on the form

$$U(z,z^*,\tau) = \int_o^\tau dt \int_{\partial D} N_o(U,S)\,ds , \tag{4.10}$$

where

$$N_o(u,v) := i\left(\tilde{P}\frac{dz^*}{ds} - \tilde{P}^*\frac{dz}{ds}\right) . \tag{4.11}$$

This in turn may be expressed in terms of tangential and normal derivatives of the data using (4.6). When S is a Green's function, then (4.10) is directly evaluated in terms of (4.5).

REFERENCES

[1] Barenblat, G., Zeltov, I., and Kochiva, I. "Basic concepts in the theory of seepage of homogeneous fluids in fissured rock", J. Appl. Math. Mech. 24(1960), 1286-1303.

[2] Bergman, S. "Integral Operators in the Theory of Linear Partial Differential Equations", Springer-Verlag, Berlin, 1961.

[3] Bhatnagar, S., and Gilbert, R.P. "Bergman type operators for pseudoparabolic equations in several space variables", Bull. Mat. Soc. Mat. Rep. Soc. Rom. 18 (66) (1974).

[4] Bhatnagar, S., and Gilbert, R.P. "A function theoretic approach to higher order pseudoparabolic equations", Math. Nachrichten (to appear).

[5] Bhatnagar, S., and Gilbert, R.P. "Constructive methods for solving $\Delta_4^2 u + A\Delta_4 u_t + Bu_t + C\Delta_4 u + Du = 0$", Math. Nachrichten (to appear).

[6] Brown, P.M. "Integral operators for fourth order linear parabolic equations", SIAM J. Math. Anal. (to appear).

[7] Brown, P.M., and Gilbert, R.P. "Constructive methods for higher order, analytic, Sobolev-Galpern equations", Bulletin Mathématique (to appear).

[8] Brown, P.M., Gilbert, R.P., and Hsiao, G.C. "Constructive function theoretic methods for fourth order pseudoparabolic equations in two space variables", Rendiconti di Matematica (to appear).

[9] Colton, D.L. "Bergman operators for parabolic equations in two space variables", Proc. Amer. Math. Soc. 38(1970), 119-126.

[10] Colton, D.L. "Integral operators and the first initial boundary value problem for pseudoparabolic equations with analytic coefficients", J. Diff. Equats. 13(1973), 506-522.

[11] Colton, D.L. "On the analytic theory of pseudoparabolic equations", Quart. J. Math. 23(1972), 179-192.

[12] Rundell, W. "The solution of the initial-boundary value problems for pseudoparabolic partial differential equations", Proc. Royal Soc. Edinburgh (to appear).

[13] Rundell, W., and Stecher, M. "A method of ascent for parabolic and pseudoparabolic partial differential equations", SIAM J. Math. Anal. (to appear).

[14] Showalter, R.E. "Partial differential equations of Sobolev-Galpern type", Pac. J. Math. 31(1969, 789-794.

[15] Showalter, R.E.
"Well-posed problems for a partial differential equation of order 2m+1", SIAM J. Math. Anal. 1(1970), 214-231.

[16] Showalter, R.E., and Ting, T.W.
"Pseudoparabolic partial differential equations", SIAM J. Math. Anal. 1(1970), 1-26.

[17] Sobolev, S.
"Some new problems in mathematical physics", Izv. Akad. Nauk. SSSR Ser. Mat. 18(1954), 3-50 (Russian).

[18] Stewartson, K.
"On the stability of a spinning top containing liquid", J. Fluid Mech. 5(1959), 577-592.

[19] Taylor, D.W.
"Research on Consolidation of Clays", MIT Press, Cambridge, 1942.

[20] Ting, T.W.
"Parabolic and pseudoparabolic partial differential equations", J. Math. Soc. Japan 21(1969), 440-453.

[21] Ting, T.W.
"Certain non-steady flows of second order fluids", Arch. Rat. Mech. Anal. 14(1963), 1-26.

[22] Vekua, I.N.
"New Methods for Solving Elliptic Equations", John Wiley, New York, 1967.

[23] Wedemeyer, E.H.
"The unsteady flow within a spinning cylinder", J. Fluid Mech. 20(1964), 383-391.

[24] Wedemeyer, E.H.
"Dynamics of liquid-filled shell ; theory of viscous corrections to Stewartson's stability problem", BRL Report 1287 (1965).